Algebra with algebra tiles: What's the point?

Marc North explores how algebra tiles, together with deliberate procedural and conceptual variation, can make algebra more meaningful, conceptual and less cognitively overloading.

Why is this stuff so hard?

I think many students find algebra difficult. Perhaps one of the reasons for this is that, like many other areas of the mathematics curriculum, it is not always immediately obvious how algebraic ideas link to everyday experiences and understandings that we are familiar with. In other words, the mathematics does not have an external reference point in the real world beyond the realm of mathematics. From a cognitive learning perspective (see, for example, Willingham, 2009), we learn by connecting new learning to existing and familiar experiences and understandings and these connections help us memorise and then remember things - which is partly why helping students to develop relational understanding in mathematics is so important. For example, when needing to explain a complex or abstract idea to someone else, one strategy is to draw on a concrete example to connect the idea to. This lowers the abstractness of the idea by linking it to something familiar. This creates a challenge when learning algebra because it exists in an abstract space largely disconnected from real-life experiences and familiar understandings. This makes it harder to reduce the level of abstraction by concretising algebraic concepts, which in turn makes it more likely for cognitive overload to occur. Arguably, then, a key task of a mathematics teacher is to give students access to abstract ideas by lowering (initially, at least) the abstractness of the ideas, by connecting the ideas to familiar experiences, ideas and understandings, and by facilitating concrete and physical experiences of the ideas, sometimes captured in pictures and manipulatives. In the discussion below, my aim is to share some personal learning experiences of how algebra tiles can do this; and, also, how careful consideration of procedural and conceptual variation when using algebra tiles (or any manipulative or representation) can support the type of relational understanding that leads to deep conceptual learning and stronger memorisation.

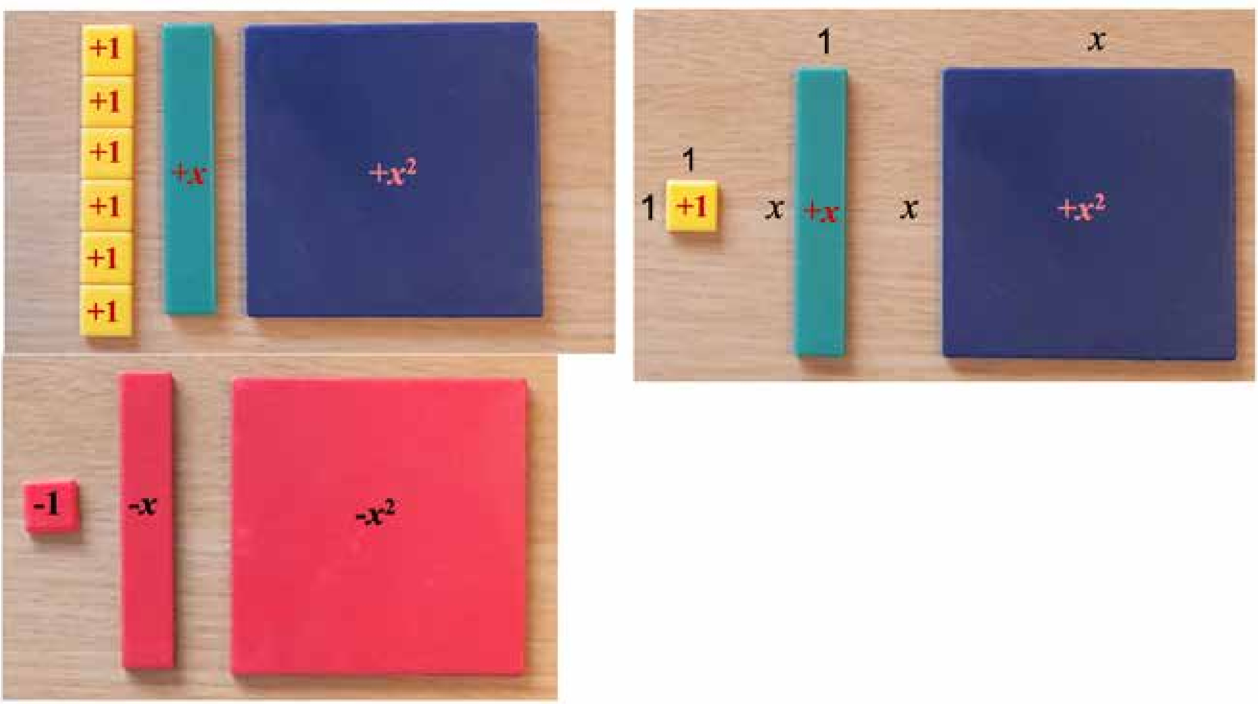


Figure 1: Algebra tiles. Note The algebra tile pictures drawn here are not to scale simply due to space restrictions. When using pictures like these in a classroom, it would be helpful if they were drawn to scale so that students can check the relationships between the tiles.

Some ground rules for algebra tiles

Algebra tiles are a collection of square and rectangular tiles that represent both numeric and abstract algebraic terms. The three tiles are based on an *area model* understanding of a number, such that the product of the factors (the dimensions of the tiles) gives the value of the number (the area of the tiles).

Crucially, the '*x*' dimension is not a multiple of 1, or else the temptation is to attach a fixed value for *x*. This is done deliberately to challenge the misunderstanding that some students have that the main goal in algebra is to find a specific value for the variable(s). Negative versions on the reverse side of the tiles and coloured red … leading to the potentially inevitable question of how can we have negative dimensions and negative areas? (See Figure 1)

Here is the first key learning point I want to highlight. Namely, that even though manipulatives (and representations) can help to concretise abstract ideas, both still require a degree of abstraction on the part of the user that sometimes does not link in a logical way to a familiar experience or common-sense way of thinking. This is because mathematics is a unique and self-contained language, with its own internal rules and procedures; and, as such, it is not possible to link all aspects of mathematics to extra-mathematical experiences. In addition, the algebra tiles embody a specific understanding of 'number-as-area' as opposed to 'number-as length' (which would be better represented on a numberline), and, in so doing, represent a very particular but also limited perspective. Taken together, then, the above points highlight that every representation – whether it be concrete (manipulative), picture, numeric or symbolic, has limitations and embodies a particular understanding of a concept. However, exploring concepts through the lens of varied representations provides the potential for a more holistic and deeper understanding.

Mathematics time

For all of the activities below, have a go at the activities yourself either with actual algebra tiles or by drawing pictures of the tiles to give you an experience of the key learning points that will be discussed. We are also going to make a deliberate pedagogic move by starting with photos of actual algebra tiles and then shifting into using drawn pictures of the tiles. This will demonstrate that, even if manipulatives are not available, picture representations can also help to concretise abstract ideas.

When using the activity in Figure 2 with other groups, most people tend to build on the existing $x + 1$ expression by adding another 1 tile. In other words, they connect their new learning to their existing understanding, and the manipulatives (and/or representation) facilitate these connections and this way of thinking. This is the 2nd key learning point. By contrast, when working purely symbolically

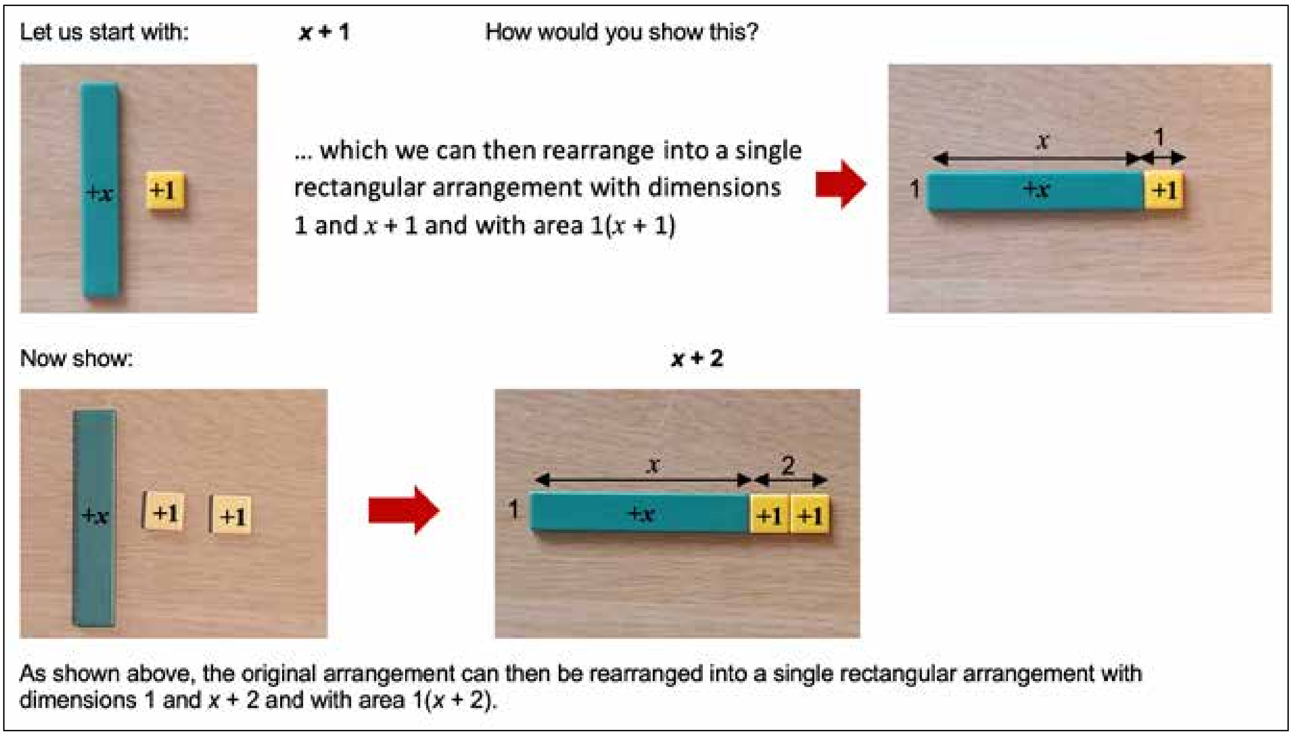


Figure 2:

with algebraic expressions, our experience is that students treat each expression as separate entities and reconstruct each expression from scratch.

Reflection prompt: When representing x + 2, did you just add another 1 to the previous x + 1 expression or did you start completely from scratch and pull out a new x tile and two new 1's tiles?

See figure 3. Once again, experience tells me that to make this arrangement you most probably just added another two *x*-tiles to the previously constructed *x* + 2 arrangement. It's not only the manipulatives or representations that facilitate this - it's also the careful and deliberate use of *procedural variation*.

A brief detour into variation theory

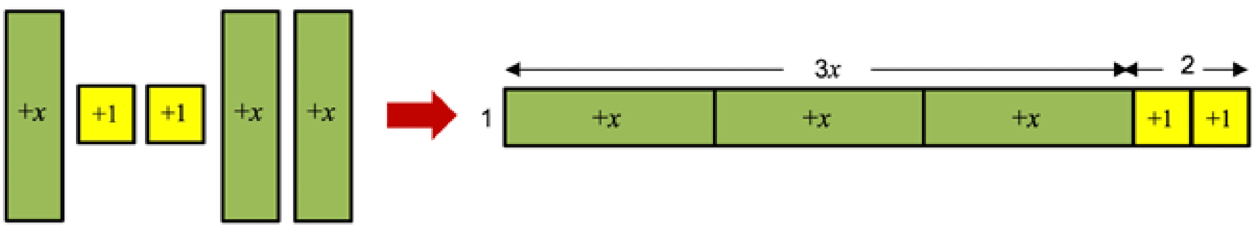
A key principle of variation theory (See, for example, Kullberg, Kempe and Marton, 2017) is that we notice differences before sameness. So, a key role of the teacher is to carefully vary certain things in the learning process to direct the students' attention to the specific aspect that they want the students to notice and learn. If students do not learn what was intended, then from a variation theory perspective this is because they have not noticed or discerned this critical learning aspect, which means that as teachers we may not have drawn enough attention to it.

There are two different aspects of the learning that can be varied. *Procedural variation* involves progressively unfolding mathematical concepts, making subtle changes that take students on a coherent journey through the formation of a concept and which emphasise connections between new and existing understandings. For example, in the algebraic expressions above, only one aspect of the expression is changed each time: *x* + 1 is changed to *x* + 2 (the constant is varied) and then *x* + 2 is changed to 3*x* + 2 (the coefficient of *x* is varied). Each of these changes focuses attention on the change from one expression to another and, more specifically, on how the expressions are connected and build on each other.

But, as was discussed above, what has made these connections more obvious is the use of varied representations: manipulatives or pictures accompany the expressions and showcase visually the structure of each expression, the change from one expression to another and, hence, the relationships between expressions. In other words, varying how we represent the expressions makes it easier to see patterns, connections and relationships. This is conceptual variation – exploring an object from different perspectives by using different representations of the object to identify and elaborate different properties of that object . Three different conceptual variations have been used thus far – symbolic representations (expressions), manipulatives (algebra tiles), and picture representations. Each conceptual variation provides a different perspective and understanding of what we are learning about, and by engaging with all three we create the potential for deeper understanding of the concept in focus.

Taken together, then, and hopefully as shown in our sequence of activities above, both procedural and conceptual variation are essential for developing deeper relational understanding of mathematical concepts – this is the 3rd key learning point. However, there is also a risk, because every time we vary something and introduce a new way of thinking about and visualising a concept, we create the opportunity for a new misunderstanding and misconception. We deal with this by being aware of the strengths and limitations of every representation we use (the 1st key learning point), and then communicating these clearly to our students.

Now show: **3*x* + 2**



Here the original arrangement can be rearranged into a single rectangular arrangement with dimensions 1 and 3*x* + 2 and with area 1(3*x* + 2)

Figure 3:

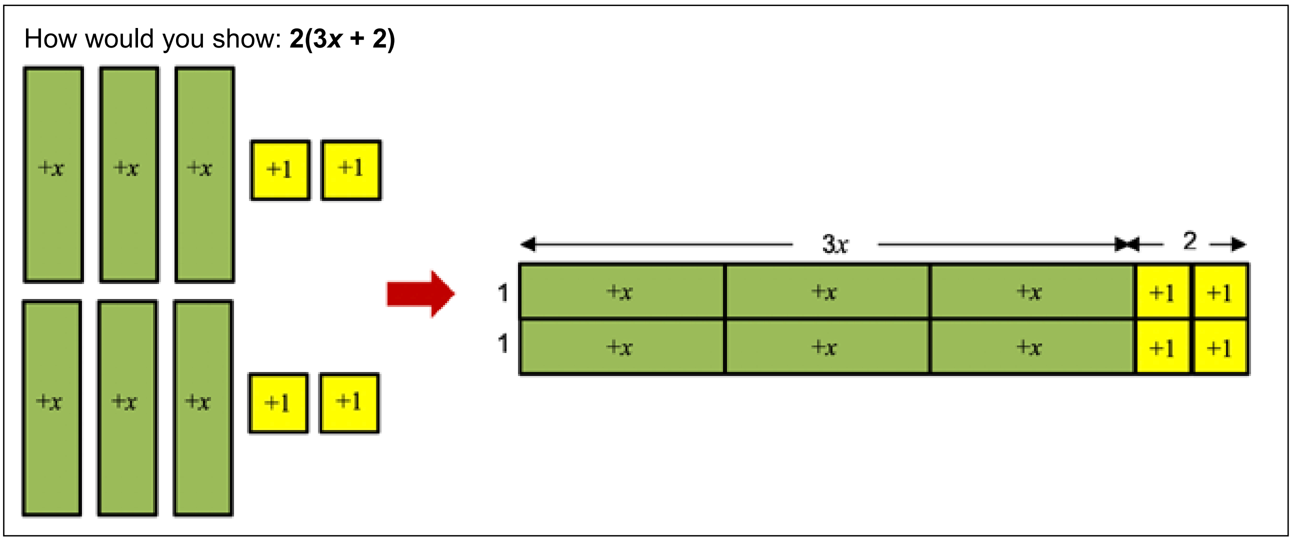


Figure 4:

Back to the mathematics

See Figure 4. As before, the original arrangement can be rearranged into a single rectangular arrangement with dimensions 2 and $3x + 2$ and with area $2(3x + 2)$. We now have a representation that reveals conceptually the relationship between $2(3x + 2)$ and $6x + 4$, which then facilitates a discussion of how to work procedurally when multiplying out $2(3x + 2)$.

Most people we have used this activity with tend to derive $2(3x + 2)$ by doubling the $3x + 2$ arrangement from the previous example – in other words, by adding in another $3x + 2$ arrangement. Once again, the subtle procedural variation focuses attention on the connection between each expression, hereby prompting a potentially different way of thinking about this expression. When thinking about this expression symbolically, many students approach the problem procedurally by trying to multiply the constant 2 into the brackets, and many make the mistake of multiplying the constant by $3x$ only and not also by the last term. By contrast, thinking about this expression conceptually as '2 lots of $3x + 2$' (which is how many people approach this problem with the algebra tiles), it's harder to make the mistake of thinking that the result must be $6x + 2$. In other words, the presence of the algebra tiles or pictures prompts conceptual thinking and ways of working (the 4th key learning point) and provides a tool to contradict and challenge common mistakes or misconceptions made when working only procedurally (the 5th key learning point).

See Figure 5. Our experience is that many people find this significantly harder to do and some resort to multiplying out the expression procedurally first to work out the expanded form $3x^2 + 2x$ and then pick and arrange tiles after-the-fact to match this solution.

There are a number of points to draw out here. First, in keeping with the previous approach, only one thing has changed from the $2(3x + 2)$ expression to the $x(3x + 2)$ expression. But, this single procedural variation has made a massive difference to the complexity of the conceptualisation of this expression. Making even a small procedural variation can significantly impact the complexity of a concept, and it's important to be aware of the conceptual demand impacted by each change so that we can support students to navigate this change - the 6th key learning point.

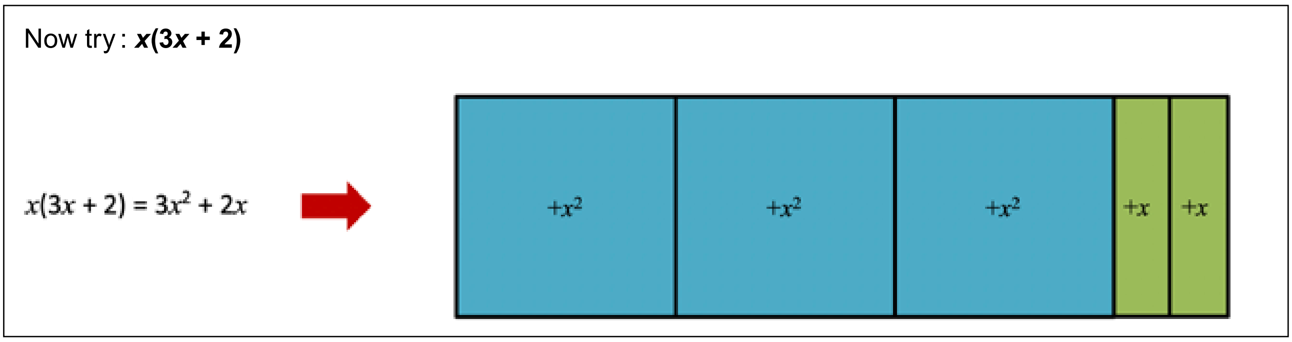


Figure 5:

Secondly, if you multiplied out the expression and then picked algebra tiles (or drew pictures) after-the-fact to match the expanded form of the expression, then the manipulatives clearly were not helpful for supporting deeper understanding. Reflecting back on the 1st key learning point, every representation has limitations and at some point a representation, including manipulatives, can stop being useful and may actually cause more confusion. It is important to know the limits of each representation, to know when a change is needed and to help students to understand this change.

Linked to the point above, I wonder if a picture is more helpful for visualising the structure of $x(3x + 2)$ than using manipulative tiles? Drawing on the growing understanding that we have tried to foreground with each previous expression that 'the structure of each expression is given by the product of the dimensions of the final rectangular arrangement', we could start with a picture of a rectangle and visualise how to partition the rectangle to match the structure of the original expression; see Figure 6.

In summary, knowing when to switch representation (i.e. conceptual variation) is an important strategy for supporting the students' understanding – the 7th key learning point.

A final example … see Figure 7. As before, the original arrangement can be rearranged into a single rectangular arrangement with dimensions $x + 1$ and $x + 2$ and with area $(x + 1)(x + 2)$. We now have a representation that reveals conceptually the relationship between $x^2 + 3x + 2$ and the product of $(x + 1)(x + 2)$, which then facilitates a discussion of the relationship between the original expression and its factorised form (and vice versa).

A couple of key points to draw out. Firstly, did you find rearranging the tiles into a single rectangular arrangement (picture on the right) a little bit trickier than before? Did you perhaps start by arranging them in one row (as we have done for all previous expressions) and then have to reconsider how to rearrange them into a single rectangle? What made things more difficult this time? In keeping with the previous approach, only one thing has changed from the $3x + 2$ expression to the $x^2 + 3x + 2$ expression with the addition of an x^2 term. But, this time, this procedural variation has conceptually altered the structure of the representation showing the factors of this expression. This explains in part why students struggle when they shift from working with expressions for linear relationships to expressions for quadratic relationships, which we may not have realised without noticing this increase in complexity in the structure of the visual arrangement. The 8th key learning point, then, is that in addition to the impact a small change in procedural variation can have on the complexity of a concept, this small change can also require a shift in how we think about the concept conceptually (i.e. conceptual variation).

Secondly, notice how quickly we have been able to progress from conceptualising the linear expression $x + 1$ to conceptualising the factor pairs for a quadratic

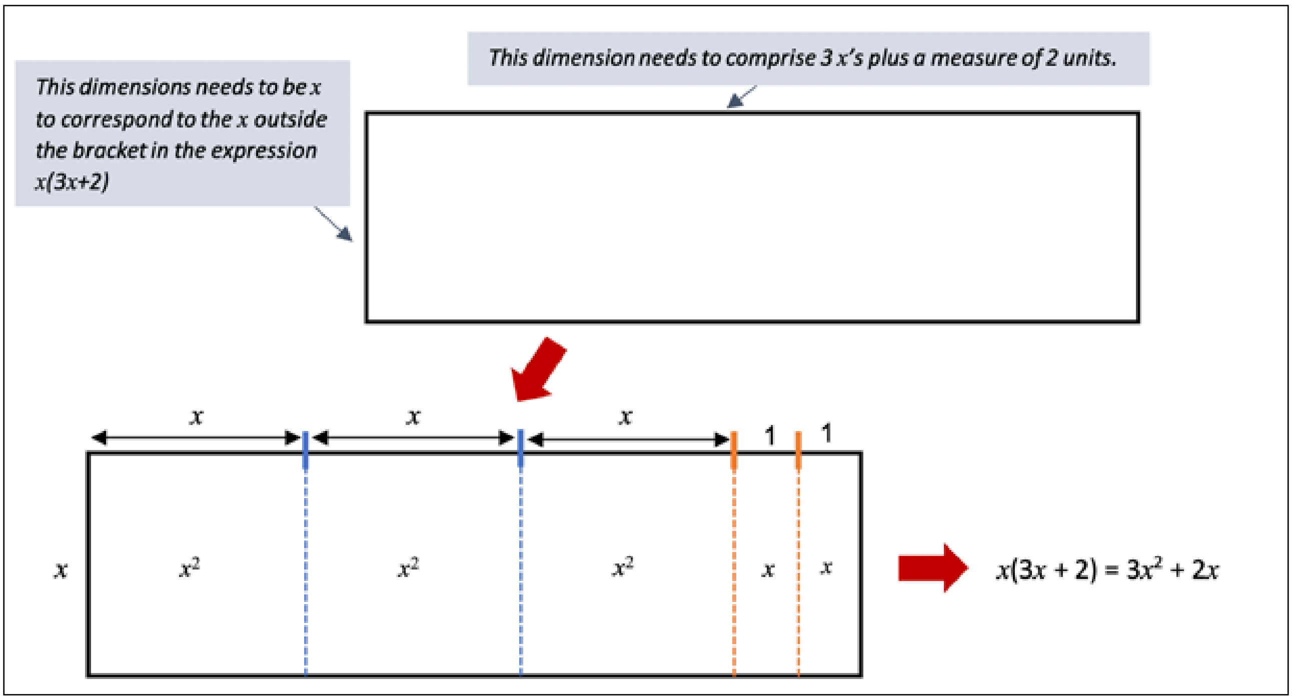


Figure 6:

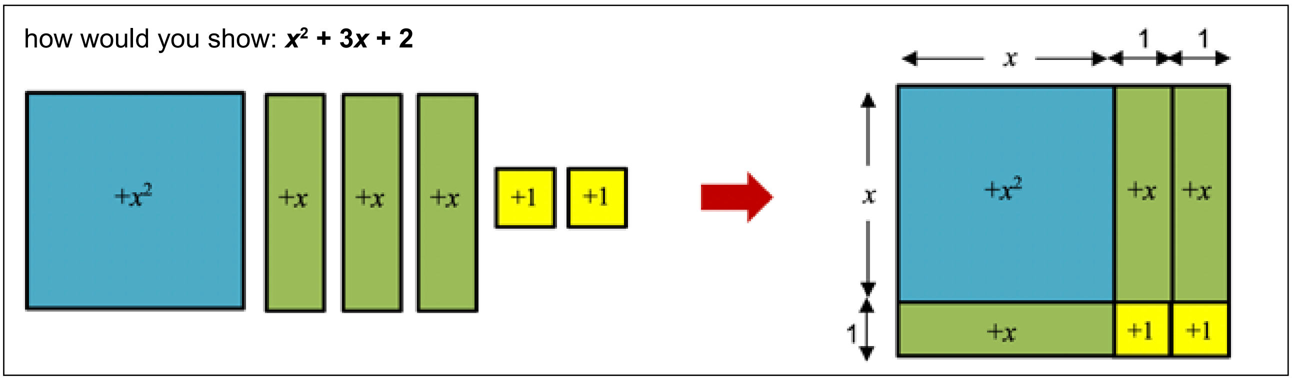


Figure 7:

expression. For me, it is the presence of the algebra tiles and/or pictures that makes this progression accessible and manageable for students. And, by using the tiles and pictures in combination with deliberate procedural and conceptual variation, students are exposed to a coherent learning journey that enables them to build connections between their existing and new understandings (2nd and 3rd key learning points) – hereby focusing their attention on relationships and structures rather than just procedures to get to an answer. Also notice how the presence of different representations – manipulative and/or pictures and symbolic notation – facilitates opportunities for opening up our classroom talk, away from mainly finding answers to having deep discussions about mathematical relationships and connections. This focus on relationships, as facilitated by the collective use of representations, open classroom discussions, and procedural and conceptual variation, is the 9th key learning point.

Making a case for the power of manipulatives and representations

Sometimes manipulatives and pictures are viewed as the terrain of primary classrooms or are reserved for children who find mathematics difficult. Yes, and as has been explored in the examples above, manipulatives like algebra tiles lower the abstractness of concepts by connecting these concepts to concrete experiences, hereby giving easier access to these concepts and reducing the risk of cognitive overload – which is particularly helpful for children who might find mathematics difficult. But, in addition, using manipulatives and pictures also shifts attention away from procedural methods for finding answers and opens up opportunities for investigating and discussing mathematical relationships and structures – all of which have the potential to foster deeper conceptual and relational understanding. This means that algebra tiles and other manipulatives are important for all students of all ages to engage with because they create the potential for deeper levels of understanding. It also makes sense that the more abstract the mathematics, the more powerful and helpful the manipulatives and representations can be for supporting understanding. With this in mind and in pursuit of goals for making mathematics more accessible and understandable, perhaps a key challenge for both primary and particularly secondary teachers, then, is to continue to experiment with even more ways to concretise abstract ideas for students and, where possible, to find appropriate manipulatives and pictures to help with this.

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References

Kullberg, A., Kempe, U. R., and Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM, 49*, 559–569. https://link.springer.com/article/10.1007/s11858-017-0858-4 .

Willingham, D. T. (2009). *Why Don't Students Like School?: A Cognitive Scientist Answers Questions About How the Mind Works and What It Means for the Classroom.* San Francisco, CA: John Wiley & Sons.

Helpful resources

Variation Unplugged with Anne Watson. https://www.youtube.com/watch?v=bfg_7pViWCE

NCETM Professional Development paper-based resources on Algebra Tiles. https://www.ncetm.org.uk/media/8d84e790f22943a/ncetm_ks3_representations_algebra_tiles.pdf